Ball Packing

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About article

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Ball Packings

and

Lorentzian Discrete Geometry

by

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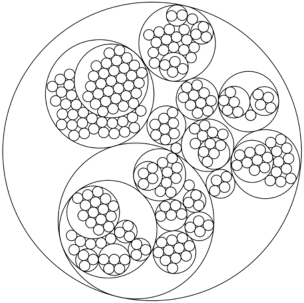
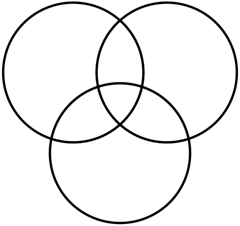
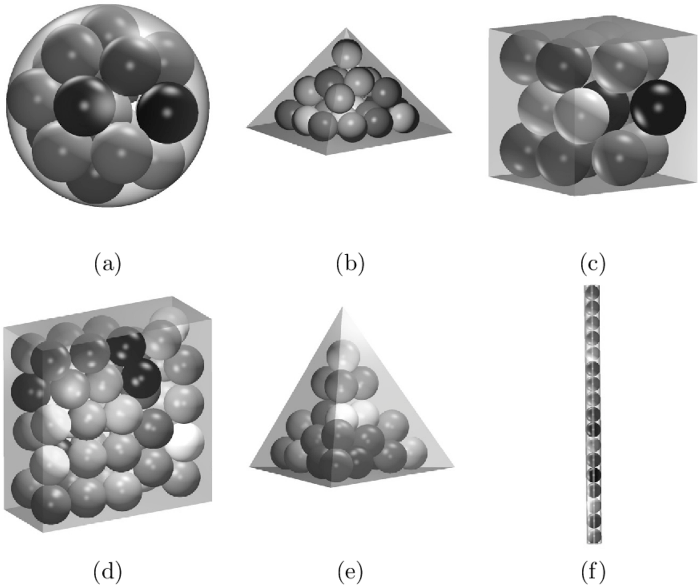
CHEN, Hao

Circle Packing





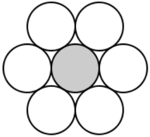
Ball (sphere) Packing

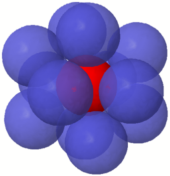


Major research problems

The kissing number problem

The kissing number problem asks for the number of non-overlapping unit balls that can be arranged to touch a fixed unit ball





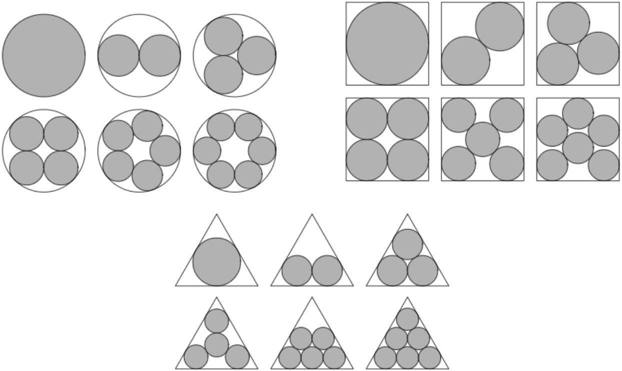
The kissing number problem

Dimension	Lower bound	Upper bound
1	2	
2	6	
3	12	
4	24[7]	
5	40	44
6	72	78
7	126	134
8	240	
9	306	363
10	500	553
11	582	869
12	840	1,356
13	1,154[13]	2,066
14	1,606[13]	3,177
15	2,564	4,858
16	4,320	7,332
17	5,346	11,014
18	7,398	16,469
19	10,668	24,575
20	17,400	36,402
21	27,720	53,878
22	49,896	81,376
23	93,150	123,328
24	196,560	

Major research problems

The densest packing problem

The densest packing problem asks for the largest density of a congruent ball packing (the balls have the same radius) in a given shape (dimension).



We work in the d-dimensional extended Euclidean space $\hat{\mathbb{R}}^d = \mathbb{R}^d \cup \{\infty\}$. Let $\|\cdot\|$ denote the Euclidean norm, and $\langle \cdot, \cdot \rangle$ denote the Euclidean inner product.

Definition 1.1.1. A *d-ball* of *curvature* κ is one of the following sets:

- $\{\mathbf{x} \in \hat{\mathbb{R}}^d \mid \|\mathbf{x} - \mathbf{c}\| \leq 1/\kappa\}$ if $\kappa > 0$;
- $\{\mathbf{x} \in \hat{\mathbb{R}}^d \mid \|\mathbf{x} - \mathbf{c}\| \geq -1/\kappa\} \cup \{\infty\}$ if $\kappa < 0$;
- $\{\mathbf{x} \in \hat{\mathbb{R}}^d \mid \langle \mathbf{x}, \hat{\mathbf{n}} \rangle \geq b\} \cup \{\infty\}$ if $\kappa = 0$.

In the first two cases, the point $\mathbf{c} \in \mathbb{R}^d$ is called the *center* of the ball, and $1/\kappa$ is the *radius* of the ball. In the second case, the ball is considered to have negative radius. In the last case, a closed half-space is considered as a ball of infinite radius, the unit vector $\hat{\mathbf{n}}$ is called the *normal vector*, and $b \in \mathbb{R}$. The boundary of a d-ball is called a *$(d-1)$-sphere*. It is a $(d-1)$-dimensional hyperplane if the ball is of curvature 0. In this thesis, balls and spheres are denoted by sans-serif upper-case letters, such as B and S.

Ball Packing

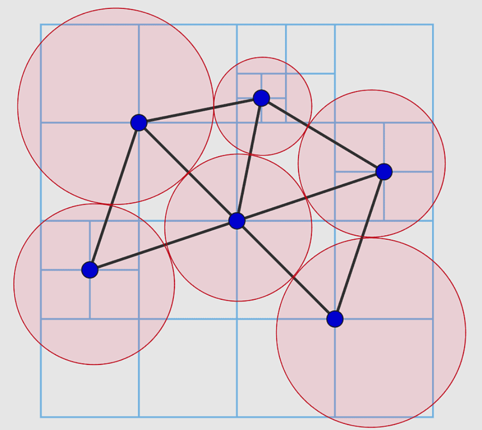
Definition

A **d-ball packing** is a collection of d-balls in $\hat{\mathbb{R}}^d$ with disjoint interiors

Ball Packing

Definition

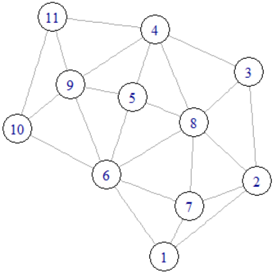
For a ball packing $\mathcal{B}$, its **tangency graph** $G(\mathcal{B})$ takes the balls as vertices, and two vertices are connected if and only if the corresponding balls are tangent

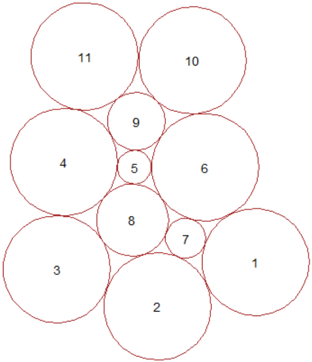


Ball Packing

Definition

A graph G is said to be **d-ball packable** if there is a d-ball packing $\mathcal{B}$ whose tangency graph is isomorphic to G. In this case, we say that $\mathcal{B}$ is a d-ball packing of G.





Previous works

Koebe–Andreev–Thurston theorem

Every connected simple planar graph is disk packable (2-ball packable).

Descartes' configurations

Descartes' theorem

Descartes' theorem states that for every four kissing, or mutually tangent, circles, the radii of the circles satisfy a certain quadratic equation. By solving this equation, one can construct a fourth circle tangent to three given, mutually tangent circles.

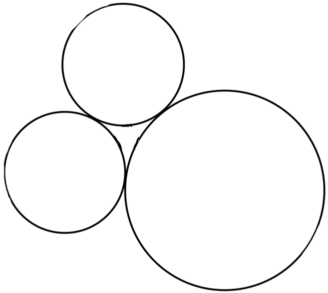
For four circles that are tangent to each other at six distinct points, with curvatures κ_i for $i = 1 \dots 4$, Descartes' theorem says:

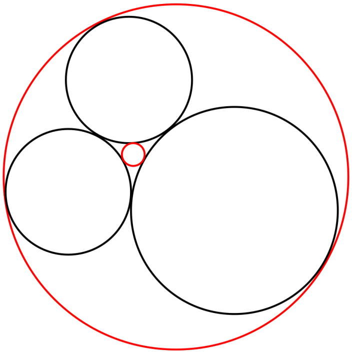
$$(\kappa_1 + \kappa_2 + \kappa_3 + \kappa_4)^2 = 2\left(\kappa_1^2 + \kappa_2^2 + \kappa_3^2 + \kappa_4^2\right).$$

To find the radius of a fourth circle tangent to three given kissing circles, the equation can be written

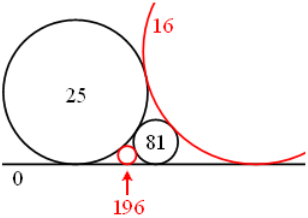
$$\kappa_4 = \kappa_1 + \kappa_2 + \kappa_3 \pm 2\sqrt{\kappa_1\kappa_2 + \kappa_2\kappa_3 + \kappa_3\kappa_1}.$$

Descartes' configurations - example





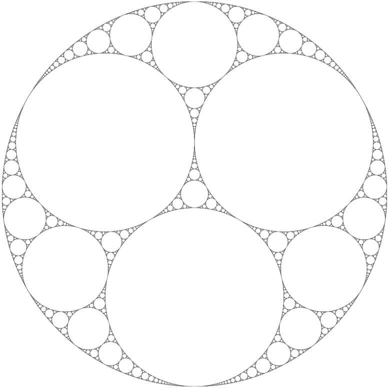
Descartes' configurations - example



Descartes' configurations - apollonian packing

Apollonian packing is created using formula:

$$\kappa_4 = \kappa_1 + \kappa_2 + \kappa_3 \pm 2\sqrt{\kappa_1\kappa_2 + \kappa_2\kappa_3 + \kappa_3\kappa_1}.$$



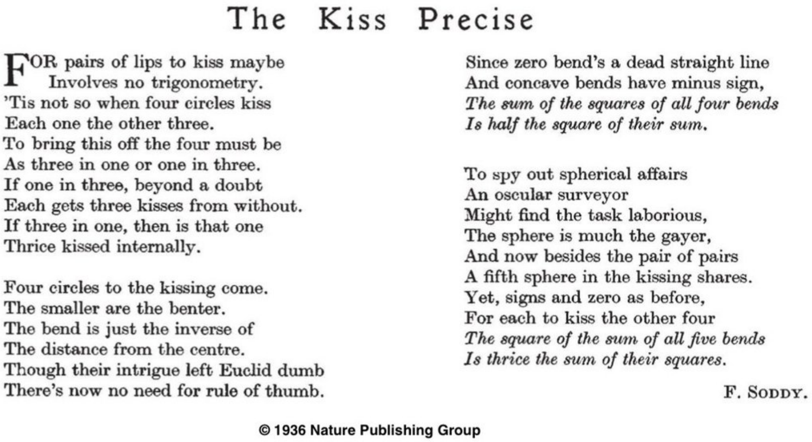
Descartes' configurations - d-dimensional

A *Descartes' configuration* in dimension d is a d-ball packing consisting of $d+2$ pairwise tangent balls. The tangency graph of a Descartes' configuration is the complete graph on $d+2$ vertices. This is the basic element for the construction of many ball packings in this thesis.

Theorem 2.1.1 (Descartes–Soddy–Gosset Theorem). *In dimension d, if a set $\mathscr{D} = \{\mathsf{B}_1, \ldots, \mathsf{B}_{d+2}\}$ of $d+2$ balls form a Descartes' configuration, are κ_i are the curvature of B_i $(1 \leq i \leq d+2)$, then*

$$\sum_{i=1}^{d+2} \kappa_i^2 = \frac{1}{d}\Big(\sum_{i=1}^{d+2} \kappa_i\Big)^2 \tag{2.1}$$

Soddy poem in Nature magazine (1936)



The Kiss Precise

FOR pairs of lips to kiss maybe
Involves no trigonometry.
'Tis not so when four circles kiss
Each one the other three.
To bring this off the four must be
As three in one or one in three.
If one in three, beyond a doubt
Each gets three kisses from without.
If three in one, then is that one
Thrice kissed internally.

Four circles to the kissing come.
The smaller are the benter.
The bend is just the inverse of
The distance from the centre.
Though their intrigue left Euclid dumb
There's now no need for rule of thumb.
Since zero bend's a dead straight line
And concave bends have minus sign,
The sum of the squares of all four bends
Is half the square of their sum.

To spy out spherical affairs
An oscular surveyor
Might find the task laborious,
The sphere is much the gayer,
And now besides the pair of pairs
A fifth sphere in the kissing shares.
Yet, signs and zero as before,
For each to kiss the other four
The square of the sum of all five bends
Is thrice the sum of their squares.

F. SODDY.

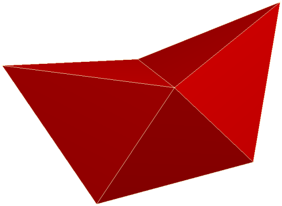
Gosset poem in Nature magazine (1937)

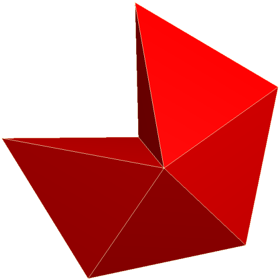
The Kiss Precise (generalized) by Thorold Gosset

And let us not confine our cares
To simple circles, planes and spheres,
But rise to hyper flats and bends
Where kissing multiple appears,
In n-ic space the kissing pairs
Are hyperspheres, and Truth declares,
As n + 2 such osculate
Each with an n + 1 fold mate
The square of the sum of all the bends
Is n times the sum of their squares.

Definition

A simplicial d-polytope is **stacked** if it can be iteratively constructed from a d-simplex by a sequence of stacking operations.





Apollonian packings and stacked polytopes

Theorem

If a disk packing is Apollonian, then its tangency graph is stacked 3-polytopal. If a graph is stacked 3-polytopal, then it is disk packable with an Apollonian disk packing.

In higher dimensions if a d-ball packing is Apollonian, then its tangency graph is stacked $(d + 1)$-polytopal.

Graphs in the form of $K_d \star P_m$

Theorem 2.2.1. *Let $d > 2$ and $m \geq 0$. A graph in the form of*

(i) $K_2 \star P_m$ *is* 2*-ball packable for any* m*;*

(ii) $K_d \star P_m$ *is* d*-ball packable if* $m \leq 4$*;*

(iii) $K_d \star P_m$ *is* not d*-ball packable if* $m \geq 6$*;*

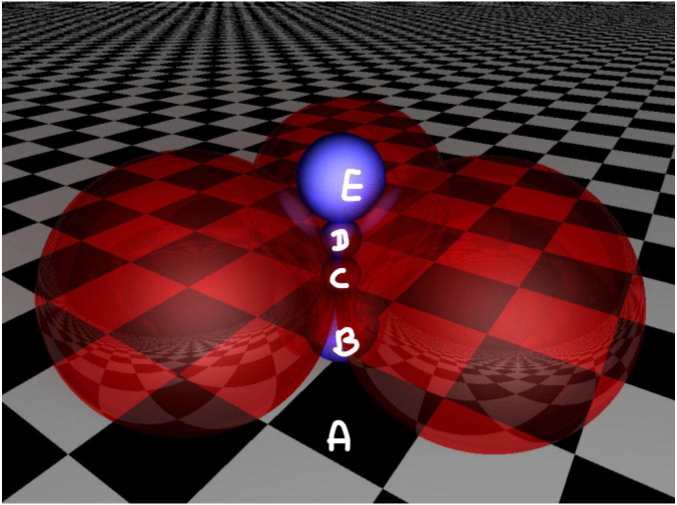
(iv) $K_d \star P_5$ *is* d*-ball packable if and only if* $d = 3$ *or* 4.

Graphs in the form of $K_d \star P_m$

(i) $K_2 \star P_m$ is 2-ball packable for any m

Graphs in the form of $K_d \star P_m$

(ii) $K_d \star P_m$ is d-ball packable if $m \leq 4$



Graphs in the form of $K_d \star P_m$

(ii) $K_d \star P_m$ is d-ball packable if $m \leq 4$
Diameter B:

$$\frac{2}{\kappa_B} = \frac{d-1}{d} < 1,$$

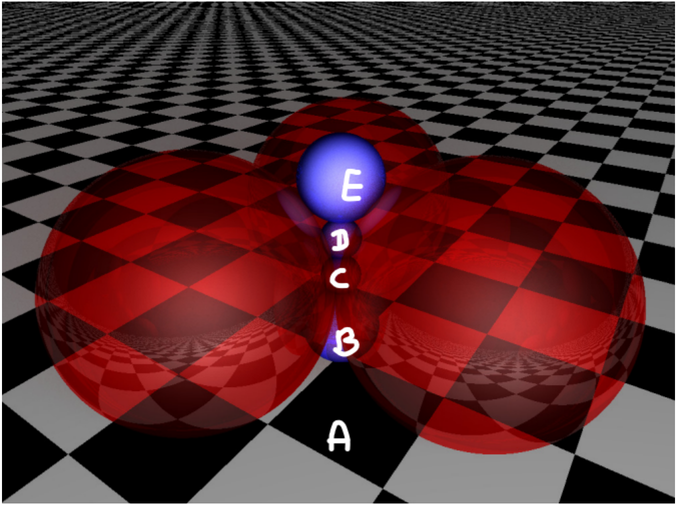
Diameter E:

$$\frac{2}{\kappa_E} = \frac{d-1}{d} < 1,$$

so diameter $B +$ diameter $E < 2$.

Graphs in the form of $K_d \star P_m$

(iv) $K_d \star P_5$ is d-ball packable if and only if $d = 3$ or 4



Graphs in the form of $K_d \star P_m$

(iv) $K_d \star P_5$ is d-ball packable if and only if $d = 3$ or 4
Diameter C:

$$\frac{2}{\kappa_C} = \frac{(d-1)^2}{d(d+1)},$$

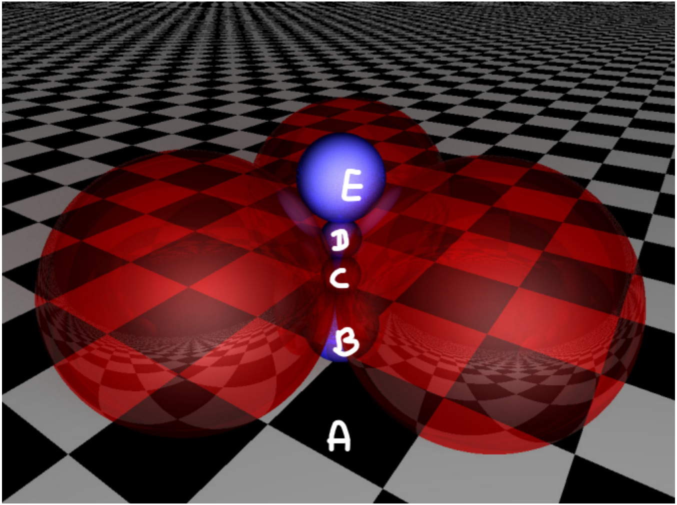
and sum of diameters B, C and E:

$$2\frac{d-1}{d} + \frac{(d-1)^2}{d(d+1)} = \frac{3d^2 - 2d - 1}{d(d+1)}$$

is < 2 only for $d \leq 4$.

Graphs in the form of $K_d \star P_m$

(iii) $K_d \star P_m$ is not d-ball packable if $m \geq 6$



Graphs in the form of $K_d \star P_m$

(iii) $K_d \star P_m$ is not d-ball packable if $m \geq 6$
Diameter D:

$$\frac{2}{\kappa_D} = \frac{(d-1)^2}{d(d+1)},$$

and sum of diameters B, C, D and E:

$$2\frac{d-1}{d} + 2\frac{(d-1)^2}{d(d+1)} = 4\frac{d-1}{d+1}$$

is < 2 only for $d < 3$.

Graphs in the form of $K_n \star G$

Definition

A d-kissing configuration is a packing of unit d-balls all touching another unit ball.

Theorem

A graph in the form of $K_3 \star G$ is d-ball packable if and only if G is the tangency graph of a $(d-1)$-kissing configuration.

For example, $K_3 \star G_{12}$ is 4-ball packable, $K_3 \star G_{24}$ is 5-ball packable.

Packing of grid graphs

Let $\mathbb{Z}$ be the set of integers. For $a \in \mathbb{Z} \cup \{\infty\}$, we use the notation

$$\mathbb{Z}_a = \begin{cases} \{k \mid 0 \leq x < a\} & \text{if } a < \infty, \\ \mathbb{Z} & \text{if } a = \infty. \end{cases}$$

For d integers $a_1, \dots, a_d \in \mathbb{Z} \cup \{\infty\}$, the *$d$-dimensional grid graph* of size $a_1 \times \cdots \times a_d$ is denoted by $Z_{a_1} \times Z_{a_2} \times \cdots \times Z_{a_d}$. It takes the points in $\mathbb{Z}_{a_1} \times \cdots \times \mathbb{Z}_{a_d}$ as vertices, and two vertices are connected whenever the corresponding points are at distance 1. The ∞ in the subscript are usually omitted. The graph $Z_{a_1} \times Z_{a_2} \times \cdots \times Z_{a_d}$ can also be regarded as the Cartesian product of d paths respectively on $a_1, \dots, a_d$ vertices. We therefore denote the path graph on n vertices by Z_n. Again ∞ in the subscript will be omitted, so Z denotes the infinite path graph. If the grid has a same size a in k coordinates, we will simply write Z_a^k for short. For example, $Z_2^2 \times Z_3^2$ denotes the $2 \times 2 \times 3 \times 3$ grid graph, Z_2^d denotes the d-dimensional hypercube graph, and Z^d denotes the d-dimensional integer lattice graph.

Packing of grid graphs - main results

Theorem

A grid graph of dimension d is d-ball packable

Theorem

Z^{d+1} is not d-ball packable.

Lorentzian space

Definition

Lorentzian n-space is the inner product space consisting of the vector space $\mathbb{R}^n$ together with the n-dimensional Lorentzian inner product.

Definition

Lorentzian inner product of two vectors $x = (x_0, x_1, ..., x_{n-1})$ and $y = (y_0, y_1, ..., y_{n-1})$ has the form

$$< x, y > = -x_0 y_0 + x_1 y_1 + ... + x_{n-1} y_{n-1}.$$

The end

Questions?