Circle graphs and monadic second-order logic

Bruno Courcelle, presented by Mikołaj Kot

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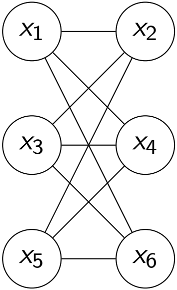

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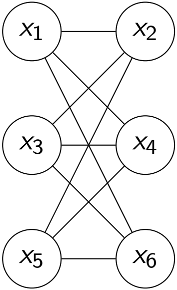

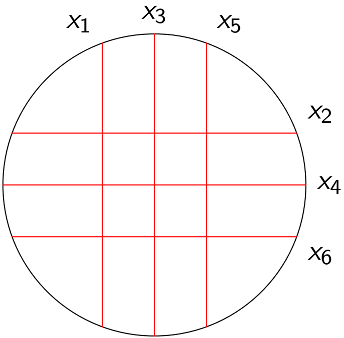

The split (join) decomposition

Split decomposition of simple graph G is a bipartition $\{A, B\}$ of V_G such that $E_G = E_{G[A]} \cup E_{G[B]} \cup (A_1 \times B_1)$ for some nonempty $A_1 \subset A$ and $B_1 \subset B$, and each of A and B has at least 2 elements

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If $\{A, B\}$ is a split then G can be expressed as the union of $G[A]$ and $G[B]$ linked by complete biparte graph.

A connected graph without split is said to be prime. Connected graphs with less than 4 vertices are thus prime.

The split (join) decomposition

Let H and K be two disjoint graphs with distinguished vertices h in H and k in K. We define $H \boxtimes_{(h,k)} K$ as the graph with set of vertices $V_H \cup V_K - \{h, k\}$ and edges $x - y$ such that, either $x - y$ is an edge of H or of K, or we have an edge $x - h$ in H and an edge $k - y$ in K.

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If $\{A, B\}$ is a split, then $G = H \boxtimes_{(h,k)} K$ where H is $G[A]$ augmented with a new vertex h and edges $x - h$ whenever there are in G edges between x and some u in B. The graph K is defined similarly from $G[B]$, with a new vertex k. These new vertices are called markers.

The split (join) decomposition

A decomposition of G is defined recursively as follows:

- $\{G\}$ is only decomposition of size 1
- if $\{G_1, \dots, G_n\}$ is decomposition of size n and $G_n = H \boxtimes_{(h,k)} K$ then $\{G_1, \dots, G_{n-1}, H, K\}$ is decomposition of size $n+1$

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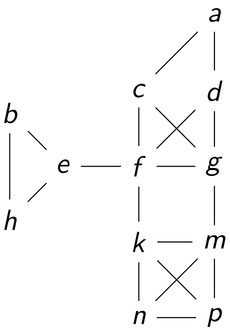
Two decompositions $\mathcal{D}$ and $\mathcal{D}'$ of a graph G are isomorphic if there exists an isomorphism of $Sdg(\mathcal{D})$ onto $Sdg(\mathcal{D}')$ which is the identity on V_G

Decomposition - example



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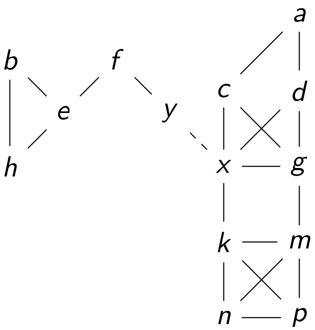

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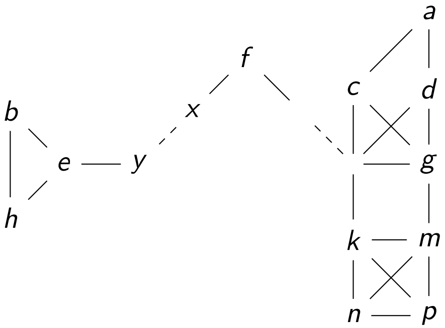

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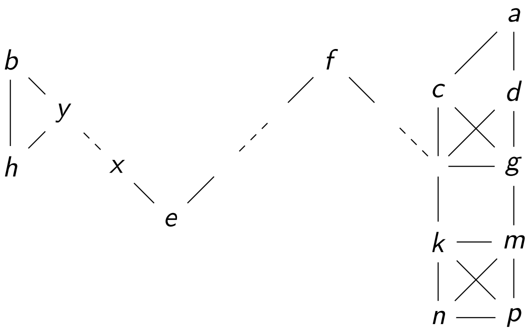

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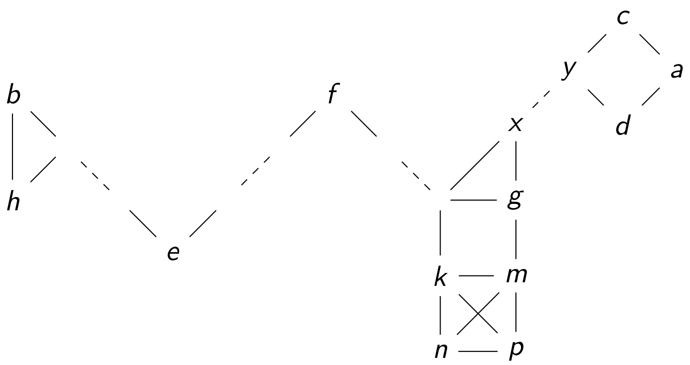

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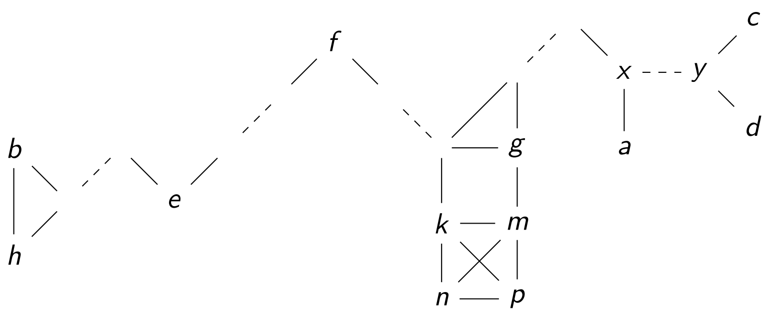

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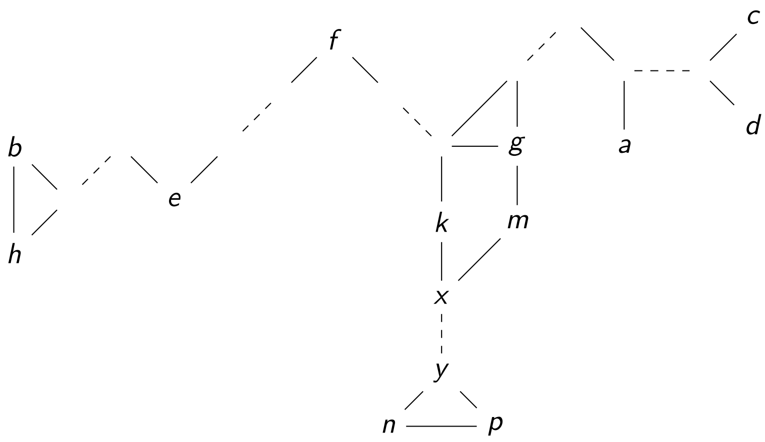

Some (not) prime graphs

All prime graphs with at least 4 vertices are 2-connected.

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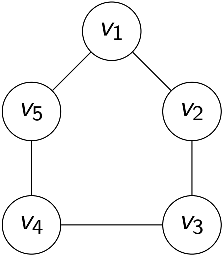

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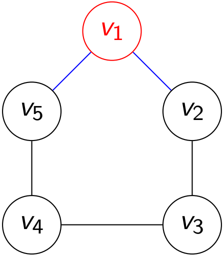

$A = \{v_1\}$ and $B = \{v_2, v_3, v_4, v_5\}$ is not a split because $|A| = 1$

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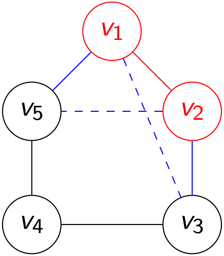

$A = \{v_1, v_2\}$ and $B = \{v_3, v_4, v_5\}$ is not a split because there is no edge $v_2 - v_5$ and $v_1 - v_3$

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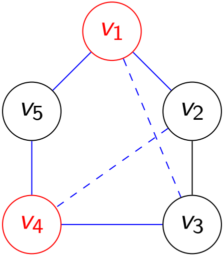

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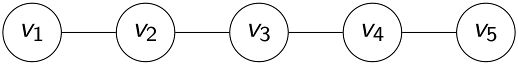

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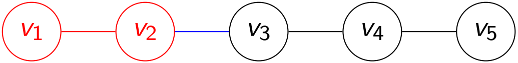

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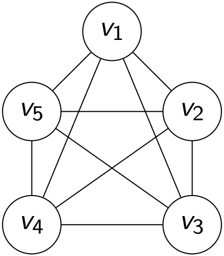

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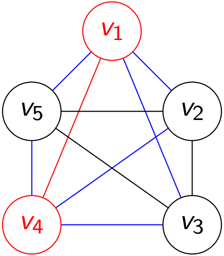

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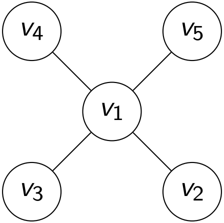

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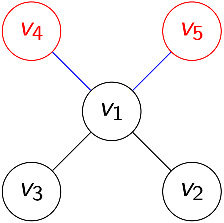

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The 2-connected undirected graphs having 4 vertices are K_4, C_4, and K_4^- (i.e., K_4 minus one edge). None of them is prime.

Canonical decomposition

A decomposition of a connected graph G is canonical if and only if:

1. each component is either prime or is isomorphic to K_n or to S_{n-1} for n at least 3
2. no two clique components are neighbour
3. the two marker vertices of neighbour star components are both centers or both not centers

Good split

A split $\{A, B\}$ is good if it does not overlap any other split $\{C, D\}$ (where we say that $\{A, B\}$ and $\{C, D\}$ overlap if the intersections $A \cap C$, $A \cap D$, $B \cap C$, $B \cap D$ are all nonempty).

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Theorem(Cunnigham, 1982) A connected undirected graph has a canonical decomposition, which can be obtained by iterated splittings relative to good splits. It is unique up to isomorphism.

Monadic second order logic

Monadic second-order logic (*MS* logic for short) is the extension of *First-order logic* by variables denoting subsets of the domains of the considered structures, and new atomic formulas of the form $x \in X$ expressing the membership of x in a set X.

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A *monadic second-order transduction* is a transformation of graphs, more generally of relational structures, expressible by MS formulas

Evaluating split decomposition graphs

For a split decomposition graph H, we let Eval(H) be the graph G defined as follows:

1. V_G is the set of vertices of H incident to no $\epsilon - edge$,
2. the edges of G are the solid edges of H not adjacent to any $\epsilon - edge$ and the edges between x and y such that there is in H a path

   $$x - u_1 - v_1 - u_2 - v_2 - \cdots - u_k - v_k - y$$

   where the edges $u_i - v_i$ are $\epsilon - edges$ and alternate with solid edges.

Evaluating split decomposition graphs

Theorem (Courcelle, 2006) If $\mathcal{D}$ is a decomposition of a connected graph G, then $Eval(Sdg(\mathcal{D})) = G$. The mapping *Eval* is an MS transduction.

Decomposition and circle graphs

Fact A graph $H \boxtimes K$ is a circle graph if and only if H and K are circle graphs. Hence, every component of the canonical split decomposition of a circle graph is a circle graph. It follows in particular that a graph is a circle graph if and only if all its prime induced subgraphs are circle graphs.

Double occurrence word

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The *alternance graph* $G(w)$, where w is double occurrence word, is undirected graph with $V(G)$ is the set of letters in w and edge $a - b$ exists if and only if $= u_1 a u_2 b u_3 a u_4 b u_5$ or $w = u_1 b u_2 a u_3 b u_4 a u_5$ for $u_1, u_2, u_3, u_4, u_5 \in A^*$

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Alternance graph is a *circle graph*

Double occurrence word - example

$w = axbcuyvbycauxv$

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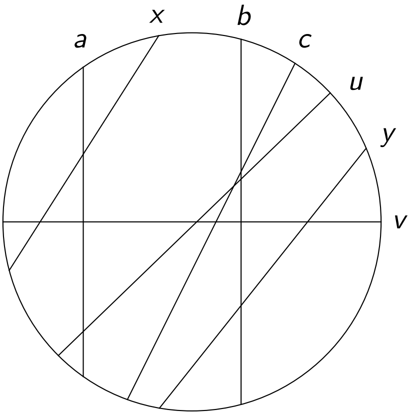

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Two equivalent words represent the same circle graph. A circle graph G is *uniquely representable* if $G = G(w) = G(w')$ implies $w \equiv w'$

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Theorem (Bouchet, 1987) A circle graph with at least 5 vertices is uniquely representable if and only if it is prime.

Double occurrence word

Theorem There exists a C_2MS transduction that associates with every prime circle graph G a double occurrence word w such that $G(w) = G$.

Vertex minors

For two sets A and B, let $A\Delta B = (A \setminus B) \cup (B \setminus A)$.
Let $G = (V, E)$ be a graph and $v \in V$ .The graph obtained by applying *local complementation* at v to G is
$G * v = (V, E\Delta\{xy : xv, yv \in E, x \neq y\})$.

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We call H is a *vertex-minor* of G if H can be obtained by applying a sequence of vertex deletions and *local complementations* to G.

Vertex minors

Theorem The set of circle graphs has a characterization in terms of three forbidden *vertex-minors*. The three forbidden *vertex-minors* are the cycles C_5, $_6$, C_7, each with one additional vertex and some edges.

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There exists C_2MS formula that checks if given graph is a circle graph, but is not constructive.

Relational structures

With $w = a_1 a_2 \dots a_{2n}$ we associate the relational structure $S(w) = \langle \{1, \dots, 2n\}, suc, slet \rangle$ where $suc(i, j)$ holds if and only if $j = i + 1$, with also $suc(2n, 1)$, and $slet(i, j)$ holds if and only if $i \neq j$ and $a_i = a_j$, and the structure $\overline{S}(w) = \langle \{1, \dots, 2n\}, \overline{suc}, slet \rangle$ where $\overline{suc} = suc \cup suc^{-1}$.

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The mapping that associates $G(w)$ with $S(w)$ is an MS transduction.

4-regular simple graphs

Lemma There exist two *MS* transductions that associate with every connected 4-regular simple graph H:

1. a set of circuits with vertex set $V_H \times \{1,2\}$, that represent all Eulerian trails of H, and
2. the structures $\langle V_H\ ,\ edg_H\ ,\ edg_G(E)\rangle$ for all Eulerian trails E of H

Neighbours

Let w be a double occurrence word such that $G = G(w)$ is prime with at least 5 vertices, let $a, b \in V(w), a \neq b$. We say that a and b are *neighbours* in w if $w \equiv abw'$ for some w' in A^*.

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For $a, b \in V_G(\subset A)$, $a \neq b$, $u, v \in A - V_G$, we let $G(a, b; u, v)$ be the graph G augmented with the path $a - u - v - b$.

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Fact $G(a, b; u, v)$ is a circle graph if and only if a, b are neighbours in G

Fact That a and b are neighbours in G is expressible by a C_2MS formula.

Clique-width and tree-width

Theorem A set of connected circle graphs has bounded *clique-width* if and only if the set of its chord diagrams has bounded *tree-width*. More precisely, there exist functions f and g such that for every double occurrence word w, $twd(S(w)) \leqslant f(cwd(G(w)))$ and $cwd(G(w)) \leqslant g(twd(S(w)))$.

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