On the equitable distribution of points on the circle

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Problem Definition

Suppose that we have a stick of unit length that represents some resource, that we want to fairly distribute between unknown amount of people. At any moment in time, the resource will be fully divided between people (there is no amount of resources not assigned to someone).

If we know the amount of people N in advance, we would give $1/N$-th of resource to everyone. Suppose that amount of people is not known beforehand. To create even distribution, it would be necessary to take a fraction of resources from everyone upon the arrival of a new person. Assuming it's a costly operation, we're only allowed to take one person's resource and divide it between these two people.

Problem Definition

Definitions

- β_N - length of the largest part after N points have been placed
- θ_N - length of the smallest part after N points have been placed
- $f_{\beta_N} = \frac{(\beta_N - \frac{1}{N})}{\frac{1}{N}} = N\beta_N - 1$
- $f_{\theta_N} = \frac{(\frac{1}{N} - \theta_N)}{\frac{1}{N}} = 1 - N\theta_N$
- $R_N = \frac{\beta_N}{\theta_N}$
- $DC(N) = \sum_{i=0}^{N-1} |\frac{1}{N} - g_i|$

Binary Splitting

Proposition

When a new recipient arrives, break the largest part into two equal pieces.

Remark

If $N = 2^j$ then $\beta_N = \theta_N = 1/N$ and $R_N = 1$
If $2^j < N < 2^{j+1}$ then $\theta_N = 1/2\beta_N$, and $R_N = 2$

Log Breaking

Proposition

When a new recipient arrives, break the largest part in the following way:

$$lg(1+\frac{1}{N}) = lg(1+\frac{1}{2N}) + lg(1+\frac{1}{2N+1})$$

$$lg(1+\frac{1}{2N}) + lg(1+\frac{1}{2N+1}) =$$

$$lg(\frac{2N+1}{2N}\cdot\frac{2N+2}{2N+1}) =$$

$$lg(\frac{2N+2}{2N}) = lg(1+\frac{1}{N})$$

Log Breaking

Definition

$U_N =< u_i >_{i=0}^{N-1}$ is the sequence of labels of the first N points on the circle, in clockwise order, starting from 0.

Observation

Let G_N be the sorted sequence of gap sizes in descending order when N points have been placed. In log breaking,

$$G_N =< lg(1+\frac{1}{N}), lg(1+\frac{1}{N+1}), ..., lg(1+\frac{1}{2N-1})$$

Log Breaking

Observation

Let G_N be the sorted sequence of gap sizes in descending order when N points have been placed. In log breaking,

$$G_N =< lg(1+\frac{1}{N}), lg(1+\frac{1}{N+1}), ..., lg(1+\frac{1}{2N-1})$$

It follows that

$$f_{\beta_N} = N \cdot lg(1+\frac{1}{N}-1) = lg(1+\frac{1}{N})^N - 1$$

$$f_{\theta_N} = 1 - lg(1+\frac{1}{2N-1})^N$$

$$R_N = \frac{lg(1+\frac{1}{N})}{lg(1+\frac{1}{2N-1})}$$

Log Breaking

Observation

$$lim_{N->\infty} N\beta_N = lg(e) = 1.44269...$$

$$lim_{N->\infty} N\theta_N = 0.721347...$$

$$lim_{N->\infty} R_N = 2$$

$$lim_{N->\infty} f_{\beta_N} = 0.44269...$$

Log Breaking

Example

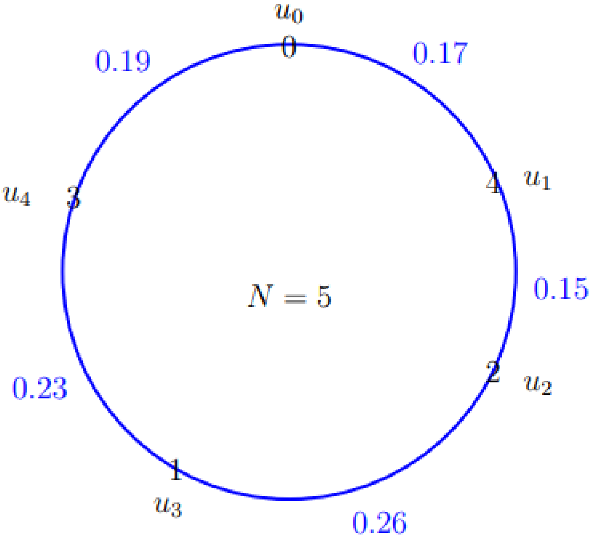
$$G_2 =< 0.58, 0.42 >$$

$$G_3 =< 0.42, 0.32, 0.26 >$$

$$G_4 =< 0.32, 0.26, 0.23, 0.19 >$$

$$G_5 =< 0.26, 0.23, 0.19, 0.17, 0.15 >$$

Log Breaking



Log Breaking

Observation

$$U_1 =< 0 >$$

$$U_2 =< 0, 1 >$$

$$U_4 =< 0, 2, 1, 3 >$$

$$U_8 =< 0, 4, 2, 5, 1, 6, 3, 7 >$$

Benford Distribution

Definition

Let $X = p_1 p_2 p_3 ... p_r$ be a positive integer written in decimal notation, then $LSD(X) = p_1$

Definition

For each N,

$$P_N = \frac{\{m_i : LSD(m_i) < p_1, i = 1, 2, ...N\}}{N}$$

then

$$P(LSD(X) < p_1) = lim_{N->\infty} P_N$$

Benford Distribution

Definition

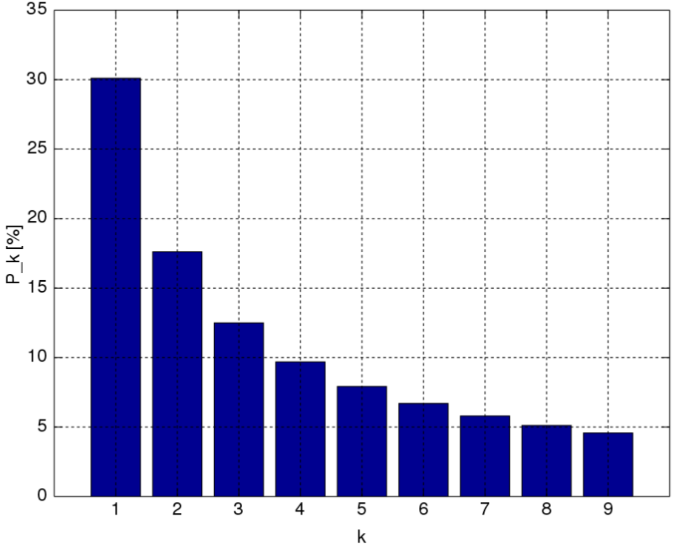
Let M be an infinite sequence of positive integers, then M is a Benford sequence iff for $X \in M$

$$P(LSD(X) < p_1) = log p_1, \text{ for each}, p_1 \in \{1, 2, ..., 10\}$$

or

$$P(LSD(X) = p_1) = log(p_1 + 1) - log p_1 = log(1 + \frac{1}{p_1}$$

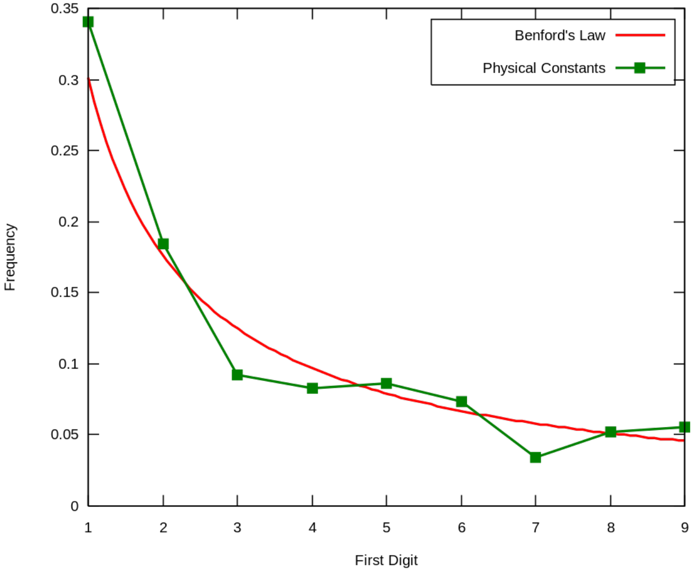
Benford Distribution



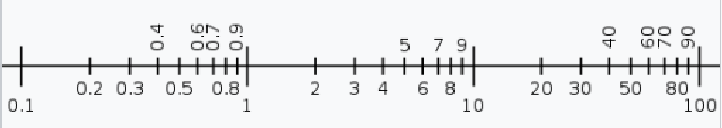
Benford Distribution

d	$P(d)$	Relative size of $P(d)$
1	30.1%	
2	17.6%	
3	12.5%	
4	9.7%	
5	7.9%	
6	6.7%	
7	5.8%	
8	5.1%	
9	4.6%	

Benford Distribution



Benford Distribution

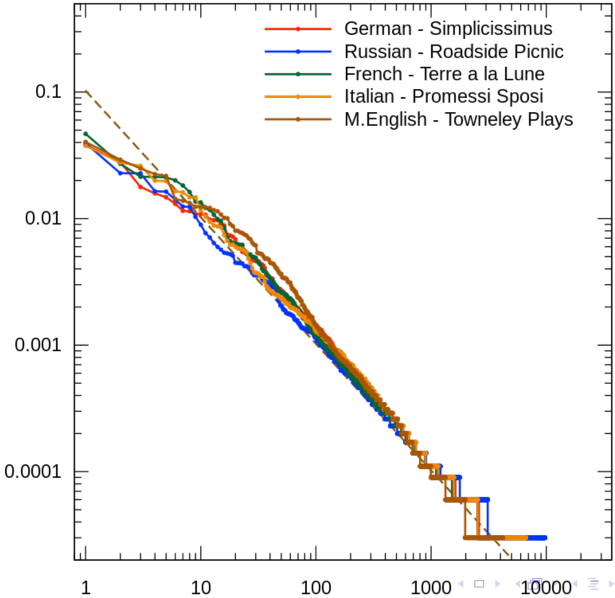


A logarithmic scale bar. Picking a random *x* position uniformly on this number line, roughly 30% of the time the first digit of the number will be 1.

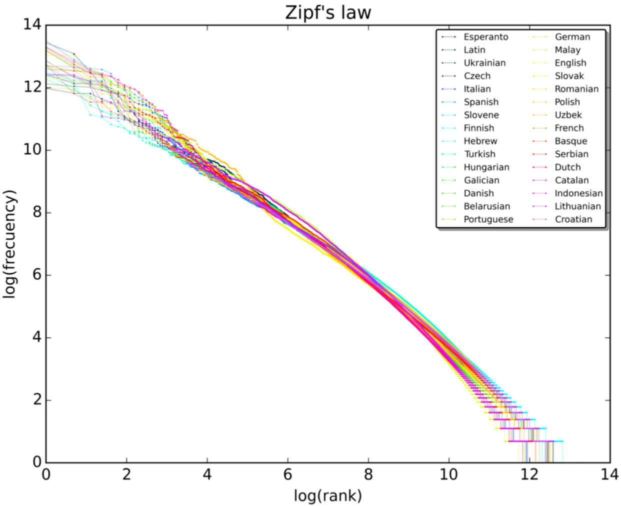
Benford Distribution

Distributions known to obey Benford's law: Fibonacci numbers, the factorials, the powers of two, and the powers of *almost* any other number. Distributions known to disobey Benford's law: the square roots and reciprocals.

Zipf's Law


German - Simplicissimus
Russian - Roadside Picnic
French - Terre a la Lune
Italian - Promessi Sposi
M.English - Towneley Plays
0.1
0.01
0.001
0.0001
1
10
100
1000
10000

Zipf's Law


Zipf's law
log(frecuency)
log(rank)
0
2
4
6
8
10
12
14
Esperanto
Latin
Ukrainian
Czech
Italian
Spanish
Slovene
Finnish
Hebrew
Turkish
Hungarian
Galician
Danish
Belarusian
Portuguese
German
Malay
English
Slovak
Romanian
Polish
Uzbek
French
Basque
Serbian
Dutch
Catalan
Indonesian
Lithuanian
Croatian

Benford Distribution

Definition

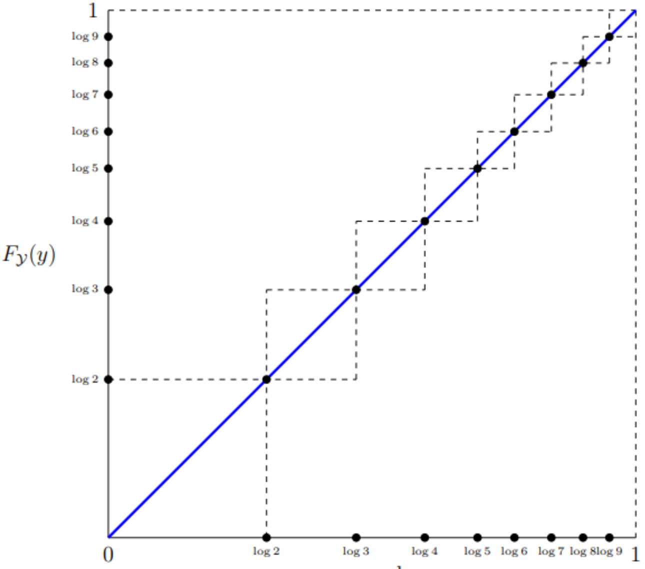
Let M be an infinite sequence of positive integers, then M is a Benford sequence iff for $X \in M$

$$P(LSD(X) < p_1) = log p_1, \text{ for each}, p_1 \in \{1, 2, ..., 10\}$$

or

$$P(LSD(X) = p_1) = log(p_1 + 1) - log p_1 = log(1 + \frac{1}{p_1})$$

Benford Distribution



Benford Distribution

Definition

Let $X = p_1 p_2 p_3 ... p_r$ be a positive integer written in decimal notation, then $LSD_r(X) = p_1 p_2 ... p_r$

Definition

Let M be an infinite sequence of positive integers, then M is a *r-Benford* sequence iff for $X \in M$

$$P(LSD_r(X) < p_1 p_2 ... p_r) = log p_1 . p_2 ... p_r = log p_1 p_2 p_r - (r - 1)$$

Golden Hops

We present a modified version of the problem: previously, the choice of new point placement was unrestricted. Now we can only choose constant C and every point is placed C units after the previous one.

Observation

If C is rational ($C = p/q$) then after placing $q + 1$ points we placed two points in the same spot, obtaining a gap of size 0. Therefore C has to be irrational.

Golden Hops

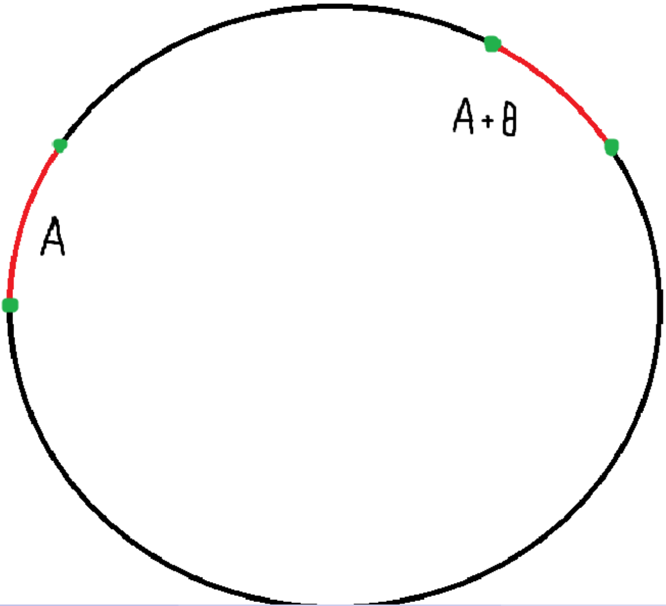
Definition

The golden section τ is a real number defined by
$\tau = (\sqrt{5} - 1)/2 = 0.618...$

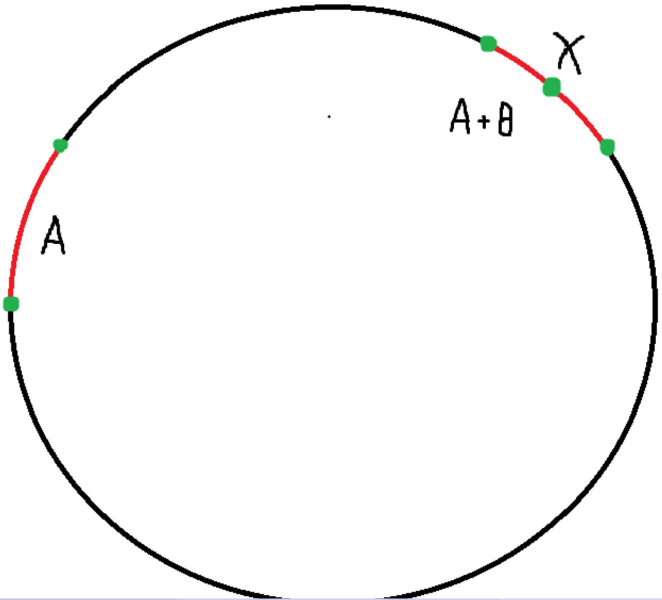
Three-gap Theorem

If C is *any* irrational number, then the gaps occur in either two or three sizes.

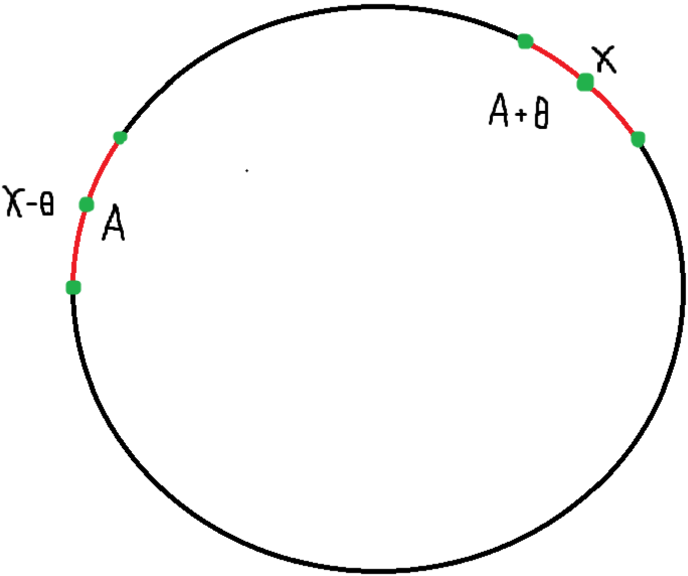
Three-gap Theorem



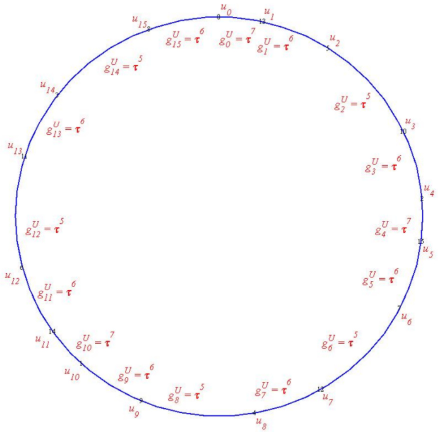
Three-gap Theorem



Three-gap Theorem

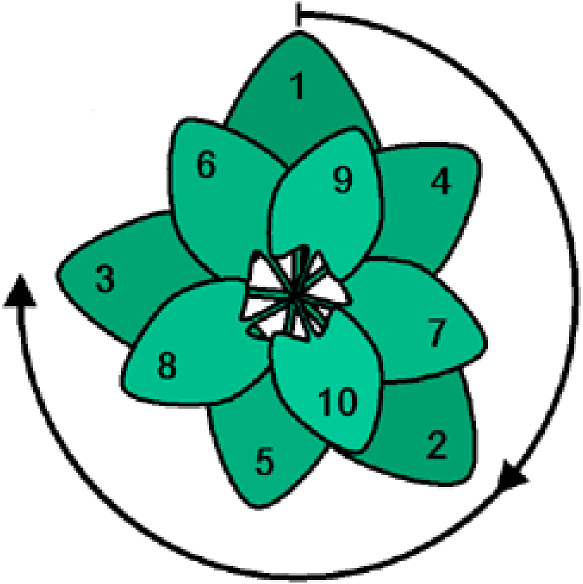


Three-gap Theorem



$$G_{16}^U = < \tau^7, \tau^6, \tau^5, \tau^6, \tau^7, \tau^6, \tau^5, \tau^6, \tau^5, \tau^6, \tau^7, \tau^6, \tau^5, \tau^6, \tau^5, \tau^6 >$$

Golden Hops



Golden Hops

Now, if $F_j < N < F_{j+1}$, where $j \geq 3$, and $Q_N(\tau^j)$ is the number of gaps of size τ^j, then:

$$Q_N(\tau^{j-2}) = F_{j+1} - N$$

$$Q_N(\tau^{j-1}) = N - F_{j-1}$$

$$Q_N(\tau^j) = N - F_j$$

Random Gaps

Random from the origin

Suppose that, we have N independent random variables, $X_1, \dots X_N$ from $\mathbb{R}^+$. We order them in an increasing sequence S, such that $X_{i+1,N} > X_{i,N}$. Let $F_h(x) = P(X_h \leq x)$ be the distribution function for $X_{h,N}$. That is, $F_h(x)$ is the probability that at least h of the random variables are less than or equal to x, hence

$$F_h(x) = \sum_{i=h}^{N} \binom{N}{i} F^i(x)(1 - F(x))^{N-i}$$

Random Gaps

Lemma

$E(X_{h,N}) = \frac{h}{N+1}$.

Let's calculate the expected value of h-th gap g_h.

$$g_h = X_{h,N} - X_{h-1,N}$$

$$\sum g_h = \sum_{i=0}^{N} X_{i+1,N} - X_{i,N} = X_{N+1,N} - X_{0,N} = 1$$

We know that the expected values of all gaps are equal, as rotating points creates the same gaps in shifted order. Therefore $E(g_h) = \frac{1}{N+1}$

Random Gaps

Lemma

$E(X_{h,N}) = \frac{h}{N+1}$.

$X_{h,N}$ can be expressed as $g_0 + g_1 + ... + g_{h-1}$.
Therefore $E(X_{h,N}) = E(g_0 + ... + g_{h-1}) = \frac{h}{N+1}$

Random Gaps

The smallest gap

Let g_S be the smallest gap.

$$P(g_S > x) = 1 - F_1(x) = (1 - Nx)^{N-1}$$

It follows that

$$E(g_S) = \int_0^{1/N} (1 - Nx)^{N-1} dx = \frac{1}{N^2}$$

The largest gap

Let g_L be the smallest gap.

$$E(g_L) = \frac{H_N}{N}$$

Results

Log breaking

$$lim_{N->\infty}\phi_N = \frac{1}{ln2} = 1.443...$$

$$lim_{N->\infty}\Omega_N = \frac{1}{2 \cdot ln2} = 0.721...$$

$$lim_{N->\infty}R_n = 2$$

Binary splitting

$$lim_{N->\infty}sup\phi_N = 2$$

$$lim_{N->\infty}inf\,\phi_N = 1$$

$$lim_{N->\infty}sup\Omega = 1$$

$$lim_{N->\infty}inf\,\Omega_N = \frac{1}{2}$$

Results

Golden Hops

$$lim_{N->\infty} sup \phi_N = \frac{1}{(\sqrt{5}\rho^3)} = 1.894...$$

$$lim_{N->\infty} inf \phi_N = \frac{1}{(\sqrt{5}\rho^2)} = 1.171...$$

$$lim_{N->\infty} sup \Omega_N = \frac{1}{(\sqrt{5}\rho)} = 0.724...$$

$$lim_{N->\infty} inf \Omega_N = \frac{1}{(\sqrt{5})} = 0.447...$$

$$R_N = \begin{cases} \frac{1}{\tau} & \text{if} N = F_j \\ \frac{1}{\tau^2} & \text{otherwise} \end{cases}$$

Results

Random

$$E(\frac{g_L}{g_S}) = N \cdot \sum_{k=1}^{N} \frac{1}{k} = N \cdot H_N$$